Improved Sample Complexity Bounds for Branch-and-Cut

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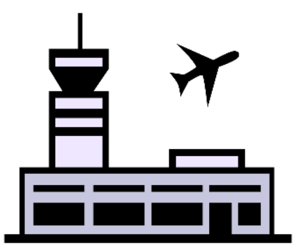
Integer programming

- Integer program (IP) in standard form:

$$\boxed{\begin{aligned} \text{Max } & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t. } & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}}$$

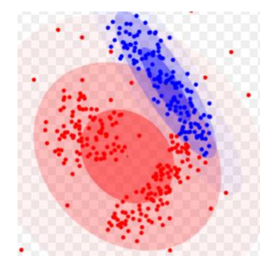
- One of the most useful and widely applicable optimization techniques



Scheduling

Routing

Combinatorial auctions



Clustering

Branch-and-cut

- Powerful tree-search algorithm used by fastest solvers to solve IPs in practice

- Our contribution: improved theory for using machine learning to tune (1) general model of tree search and (2) any-and-all aspects of branch-and-cut

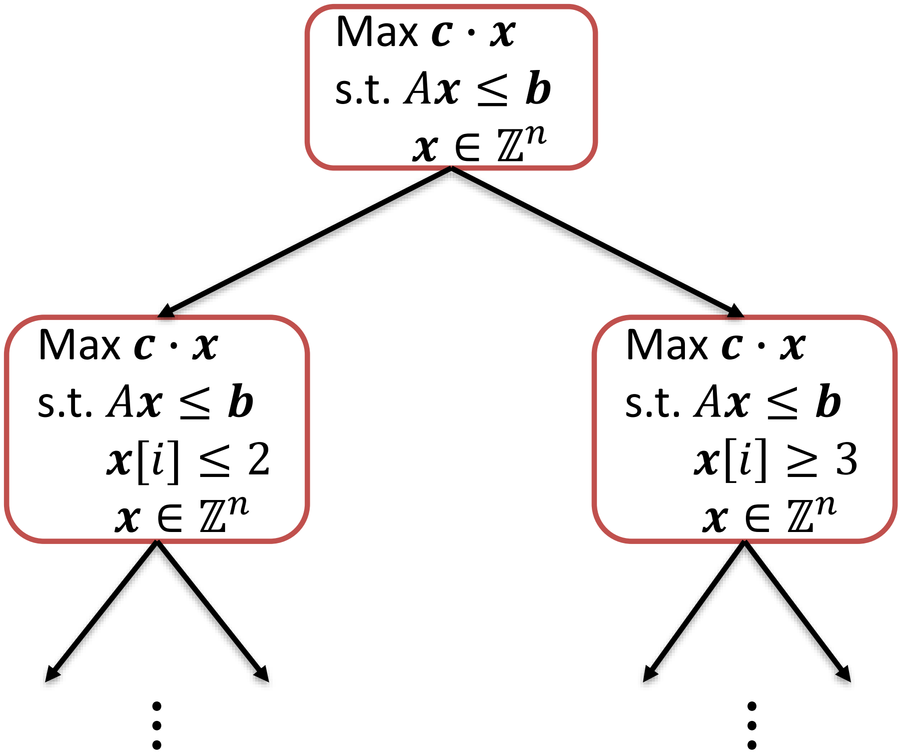
Branch-and-bound

- Powerful tree-search algorithm used to solve IPs in practice
- Uses the linear programming (LP) relaxation to do an informed search through the set of feasible integer solutions

IP		LP relaxation
Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{Z}^n$	$\longrightarrow$	Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{R}^n$

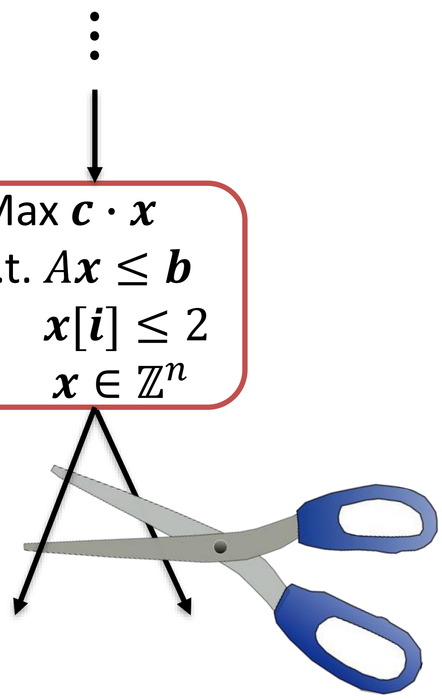
Branch-and-bound: branching

- Choose variable *i* to branch on.
- Generate one subproblem with $\boldsymbol{x}[i] \leq \lfloor \boldsymbol{x}^*_{\mathrm{LP}}[i] \rfloor$ another with $\boldsymbol{x}[i] \geq \lceil \boldsymbol{x}^*_{\mathrm{LP}}[i] \rceil$



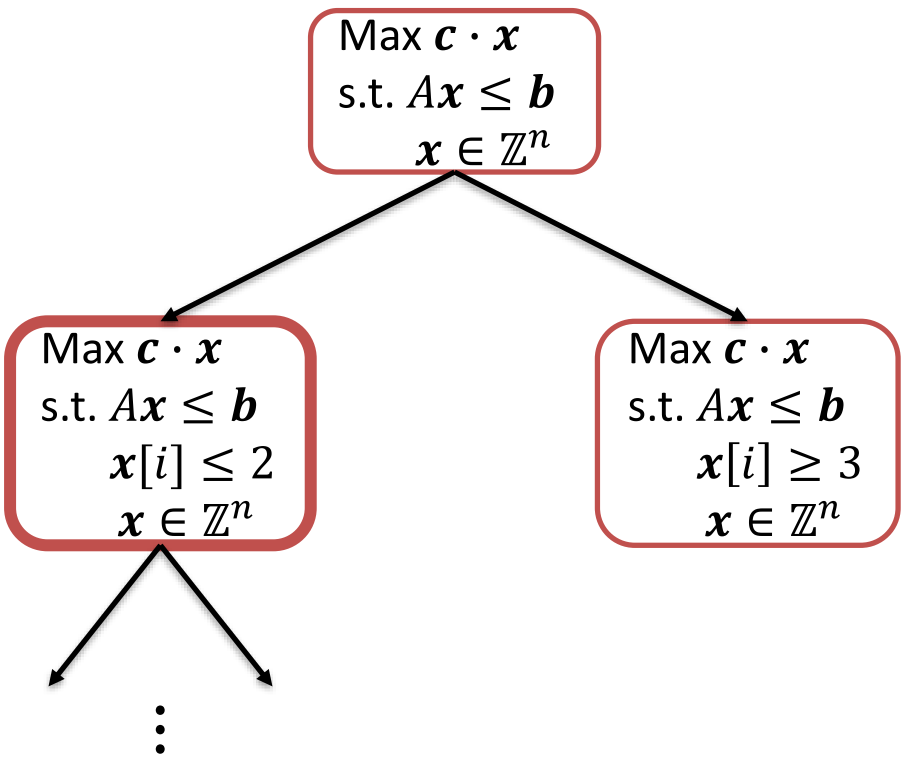
Branch-and-bound: pruning

- Prune subtrees if
  - LP relaxation at a node is integral, infeasible, or
  - (Bounding) LP optimal *worse* than best feasible integer solution found so far



Branch-and-bound: node selection

- At every stage, need to choose a leaf to explore further
- Variety of heuristics (e.g. *best-bound-first* chooses the node with the smallest LP objective)



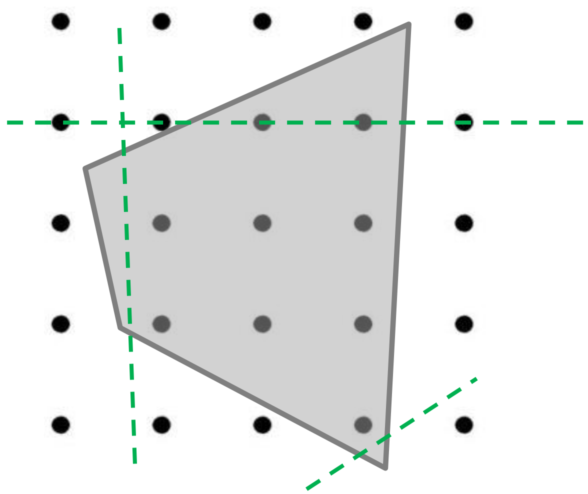
Branch-and-cut

- Branch-and-bound, but at each node may add *cutting planes*

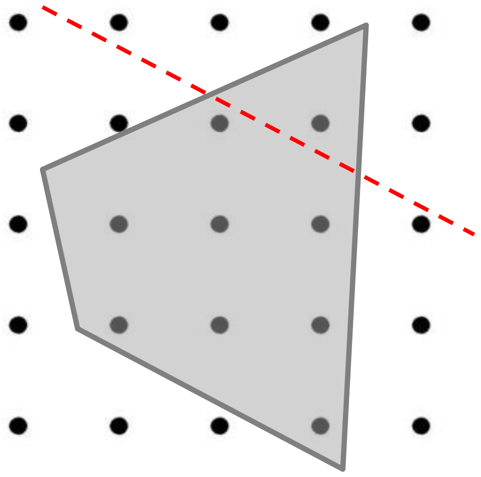
- Method of getting tighter LP relaxation bounds, and thus pruning subtrees sooner

Cutting planes

- Constraint $\boldsymbol{\alpha x} \leq \boldsymbol{\beta}$ is a *valid cutting plane* if it does not cut off any integer feasible points



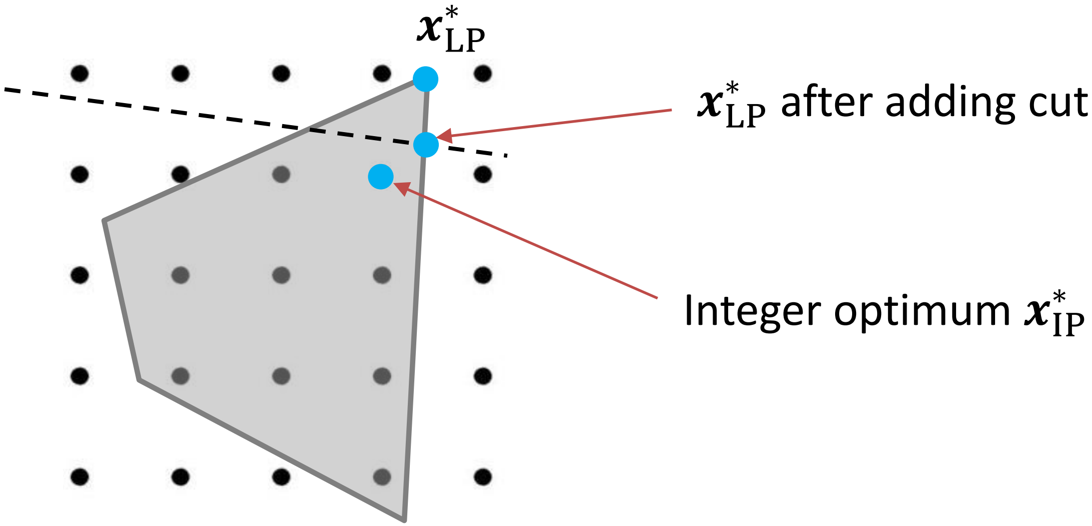
Valid cutting planes



An invalid cutting plane

Cutting planes

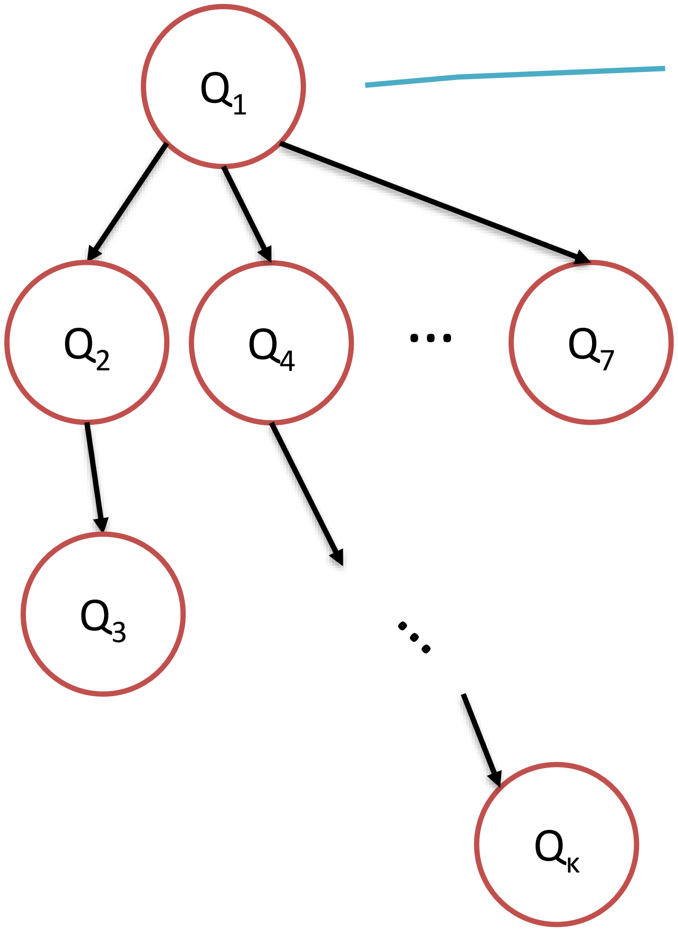
- If $\boldsymbol{\alpha x} \leq \beta$ is valid and separates the LP optimum, can speed up B&C by pruning nodes sooner



Tuning branch-and-cut

- Solvers like CPLEX, Gurobi have *numerous* parameters that control various aspects of the search (CPLEX has 170 page manual describing 172 parameters)

Abstracting away: tree search



- Select node Q that maximizes node selection rule nscore(T, Q)
  - Select action A that maximizes action score ascore(T, Q, A)
  - Either prune tree at Q, or add children
  - Continue until all nodes are pruned

Actions chosen using mixture of scoring rules:

$$\text{ascore} = \mu \cdot \text{ascore}_1 + (1 - \mu) \cdot \text{ascore}_2$$

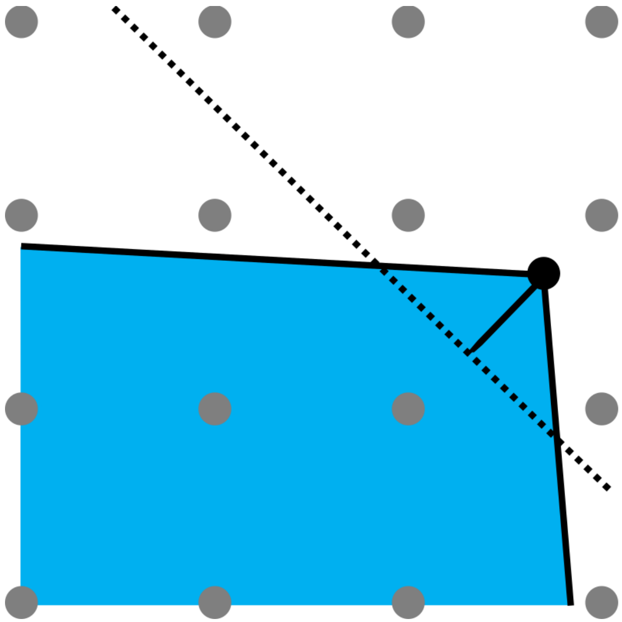
Nodes chosen using mixture of scoring rules:

$$\text{nscore} = \lambda \cdot \text{nscore}_1 + (1 - \lambda) \cdot \text{nscore}_2$$

Cut scoring rule example

Efficacy:

distance between cut and $\boldsymbol{x}_{\mathrm{LP}}^{*}$

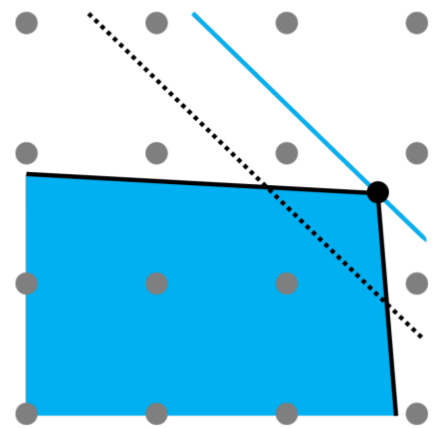


$$\mathrm{score}_1(\boldsymbol{\alpha}^T\boldsymbol{x} \leq \beta) = \frac{\boldsymbol{\alpha}\boldsymbol{x}_{\mathrm{LP}}^{*} - \beta}{\|\boldsymbol{\alpha}\|_2}$$

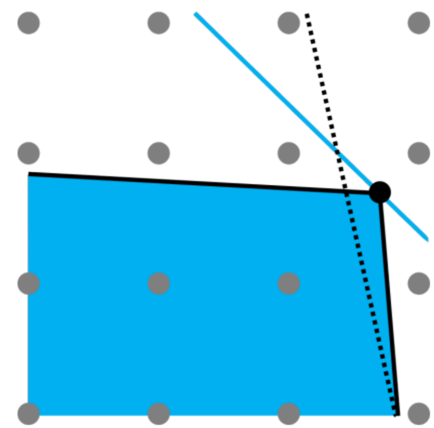
Cut scoring rule example

Parallelism:

angle between cut and objective



Better parallelism



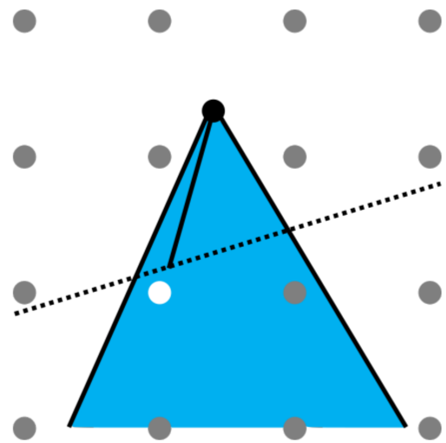
Worse parallelism

$$\mathrm{score}_2(\boldsymbol{\alpha}^T\boldsymbol{x} \le \beta) = \frac{|\boldsymbol{c}\boldsymbol{\alpha}|}{\|\boldsymbol{\alpha}\|_2\|\boldsymbol{c}\|_2}$$

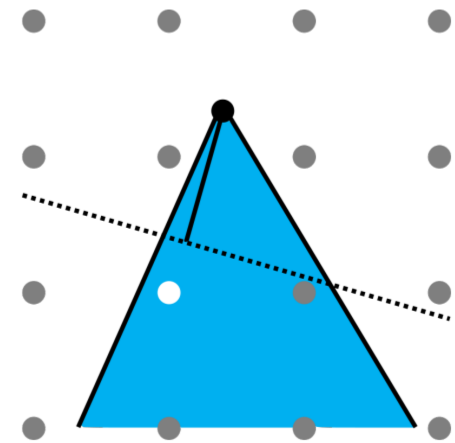
Cut scoring rule example

Directed cutoff:

distance between cut and $\boldsymbol{x}_{\mathrm{LP}}^*$, in direction of current best integer solution



Better directed cutoff



Worse directed cutoff

$$\mathrm{score}_3(\boldsymbol{\alpha}^T\boldsymbol{x} \le \beta) = \frac{\boldsymbol{\alpha}\boldsymbol{x}_{\mathrm{LP}}^* - \beta}{|\boldsymbol{\alpha}(\overline{\boldsymbol{x}} - \boldsymbol{x}_{\mathrm{LP}}^*)|} \cdot \|\overline{\boldsymbol{x}} - \boldsymbol{x}_{\mathrm{LP}}^*\|_2$$

Pathwise scoring rules

- All the previous scoring rules are *pathwise*: they only depend on the LP information accumulated along the path from the root to the node in question

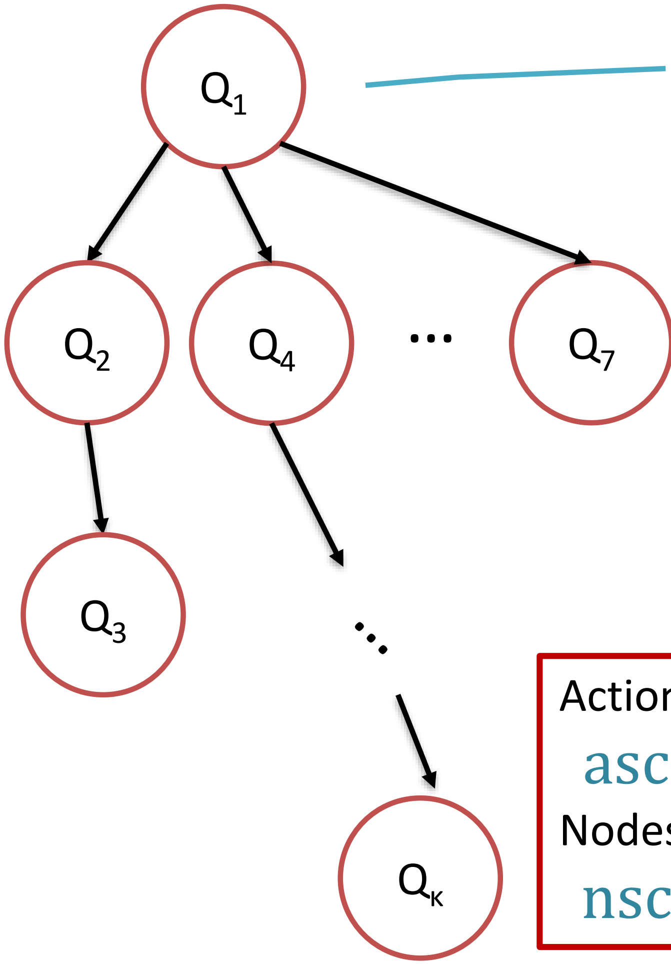
- Open source solver SCIP uses hard-coded mixture of scores to choose cuts

$$\frac{3}{5}\mathrm{score}_1 + \frac{1}{10}\mathrm{score}_2 + \frac{1}{2}\mathrm{score}_3 + \frac{1}{10}\mathrm{score}_4$$

Generalization guarantees for tree search and branch-and-cut

Distribution-dependent parameter selection of μ, λ

Parameterized tree search



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Actions chosen using mixture of **pathwise** scoring rules:

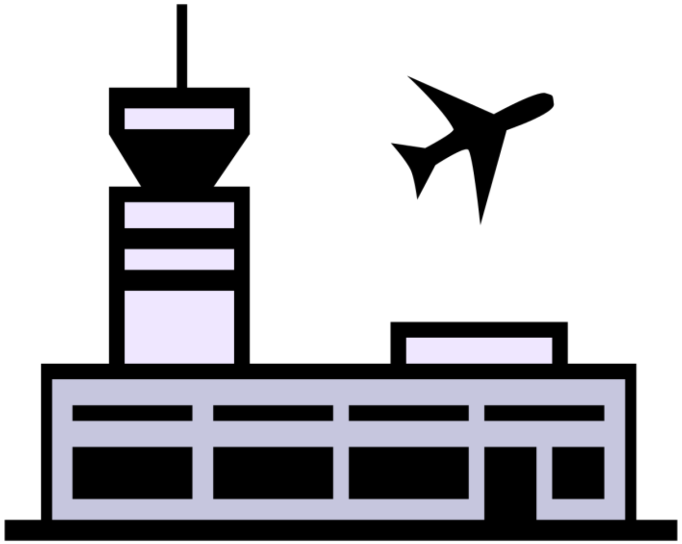
$$\text{ascore} = \boldsymbol{\mu} \cdot \text{ascore}_1 + (1 - \boldsymbol{\mu}) \cdot \text{ascore}_2$$

Nodes chosen using mixture of **pathwise** scoring rules:

$$\text{nscore} = \boldsymbol{\lambda} \cdot \text{nscore}_1 + (1 - \boldsymbol{\lambda}) \cdot \text{nscore}_2$$

Learning to tune tree search

Best parameters for airline-scheduling IPs…



…might not be useful for combinatorial-auction IPs solved by a sourcing firm

Learning to tune branch-and-cut

If a certain set of parameters yields small average branch-and-cut tree size over IP samples…

IP 1		IP N	
Max $\boldsymbol{c_1} \cdot \boldsymbol{x}$ s.t. $A_1\boldsymbol{x} \leq \boldsymbol{b_1}$ $\boldsymbol{x} \in \mathbb{Z}^n$	$\bullet\bullet\bullet$	Max $\boldsymbol{c}_N \cdot \boldsymbol{x}$ s.t. $A_N\boldsymbol{x} \leq \boldsymbol{b_N}$ $\boldsymbol{x} \in \mathbb{Z}^n$	$\sim D$

…is it likely to yield a small branch-and-cut tree on a fresh IP?

Max $\boldsymbol{c} \cdot \boldsymbol{x}$
s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$
$\boldsymbol{x} \in \mathbb{Z}^n$ $\quad \sim D$

Sample complexity

- Q – domain of input root nodes to tree search
- $F = \{f_{\mu,\lambda} : Q \to \mathbf{R} \mid \mu, \lambda\}$ class of functions (e.g. tree size)
- Sample complexity $N_F(\varepsilon, \delta)$ is the minimum $N_0 \in \mathbf{N}$ such that for any $N \geq N_0$:

$$\Pr_{Q_1,\ldots,Q_N \sim D}\left(\sup_{f \in F}\left|\frac{1}{N}\sum_{i=1}^{N} f(Q_i) - \mathbf{E}_{Q\sim D}[f(Q)]\right| \leq \varepsilon\right) \geq 1 - \delta$$

for any distribution D on Q.

Sample complexity of tuning tree search

Theorem [BPSV CP'22]: For all μ, λ, the number of samples so that the difference between average training performance and expected performance when μ, λ is used to select actions and nodes throughout the tree is (whp) at most ε is

$$\tilde{O}\left(\frac{H^2}{\varepsilon^2}(\Delta^2 \log k + \Delta \log b)\right)$$

Δ = tree depth
k = tree branching factor
b = # actions available at each node
H = cap on size of tree

First guarantee that handles multiple critical aspects of branch-and-cut:
Node selection, branching, and cutting plane selection

Generalization guarantee for tree search

Theorem [BPSV CP'22]: For all μ, λ, difference between average training performance and expected performance when μ, λ is used to select actions and nodes throughout the tree is (whp)

$$\tilde{O}\left(H\sqrt{\frac{\Delta^2 \log k + \Delta \log b}{N}}\right)$$

Δ = tree depth
k = tree branching factor
b = # actions available at each node
H = cap on size of tree

Holds for any (unknown) distribution over tree-search problem instances

First guarantee that handles multiple critical aspects of branch-and-cut:
Node selection, branching, and cutting plane selection

Tree search guarantees

- Main challenge: performance functions (e.g. size of tree) are highly discontinuous
  - Miniscule change in parameters can lead to exponential difference in tree size
- We prove that parameterized tree search is structured
- Allows us to bound the *intrinsic complexity* (pseudo-dimension from learning theory) of the class of performance functions parameterized by (μ, λ), which implies our sample complexity bounds

Tree search structure

Theorem [BPSV CP'22]:

Fix path-wise node selection scores $\mathrm{nscore}_1, \mathrm{nscore}_2$ and path-wise action selection scores $\mathrm{ascore}_1, \mathrm{ascore}_2$, and the input node Q.

There are $\leq k^{\Delta(9+\Delta)} b^{\Delta}$ rectangles partitioning $[0,1]^2$ such that for any rectangle R, the node-selection score $\lambda \cdot \mathrm{nscore}_1 + (1-\lambda) \cdot \mathrm{nscore}_2$ and action selection score $\mu \cdot \mathrm{ascore}_1 + (1-\mu) \cdot \mathrm{ascore}_2$ result in the same tree for all $(\mu, \lambda) \in R$.

Δ = tree depth
k = tree branching factor

Back to branch-and-cut

- Our result implies polynomial bounds for:
  - Branching: single-variable, multi-variable, branching on general disjunctions with bounded coefficients,…
  - Cutting planes: cover cuts, clique cuts, any cuts derived from simplex tableau (Chvátal cuts, Gomory mixed integer cuts)
  - Allows node selection to be tuned simultaneously
- Prior work
  - [Balcan et al. ICML’18] studied single-variable branching with pathwise scoring rules (our result recovers theirs)
  - [Balcan, **Prasad,** Vitercik, Sandholm NeurIPS’21] studied Chvátal cuts, but obtained a much weaker bound when these are applied throughout the tree due to not using pathwise assumption

Knapsack cover cuts – an experiment

- Set of items N, item $i \in N$ has value $p_i \geq 0$ and weight $w_i \geq 0$
- Set of knapsacks K, knapsack $k \in K$ has capacity $W_k \geq 0$
- *Goal:* find feasible packing of maximum weight

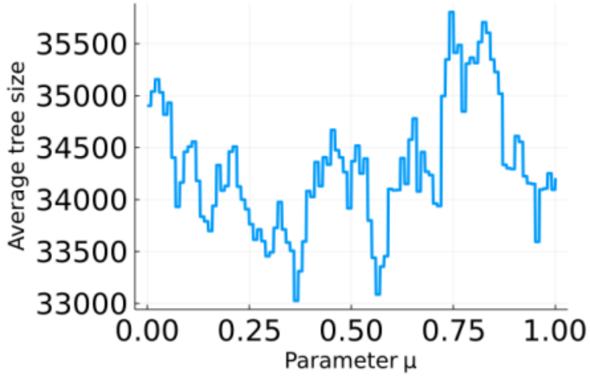
$$
\begin{array}{lll}
\text{maximize} & \Sigma_{i \in N} \Sigma_{k \in K} p_i x_{k,i} & \\
\text{subject to} & \Sigma_{i \in N} w_i x_{k,i} \leq W_k & \forall k \in K \\
 & \Sigma_{k \in K} x_{k,i} \leq 1 & \forall i \in N \\
 & x_{k,i} \in \{0,1\} & \forall i \in N, k \in K
\end{array}
$$

Knapsack cover cuts – an experiment

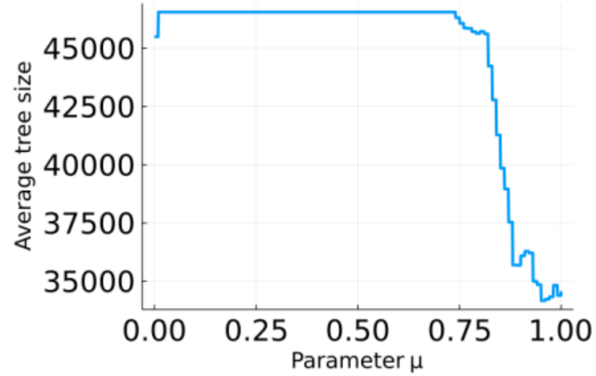
- Cover cut for knapsack k: if $w_1 + w_2 + w_3 \geq W_k$ (items 1, 2, 3 are jointly too heavy for knapsack k), can enforce the constraint $x_{k,1} + x_{k,2} + x_{k,3} \leq 2$
- We tune convex combinations of cut scoring rules to control the addition of cover cuts* throughout the branch-and-cut tree

*actually a special kind of cover cut: *extended minimal cover cuts*

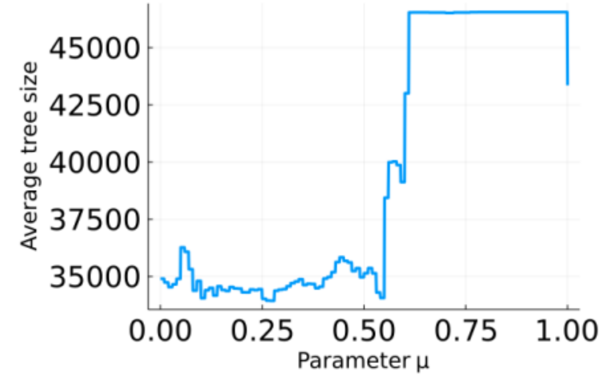
Knapsack cover cuts – an experiment



(a) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{P}$.

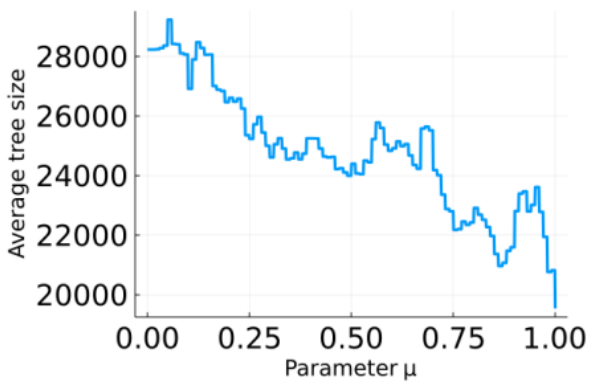


(b) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{D}$.

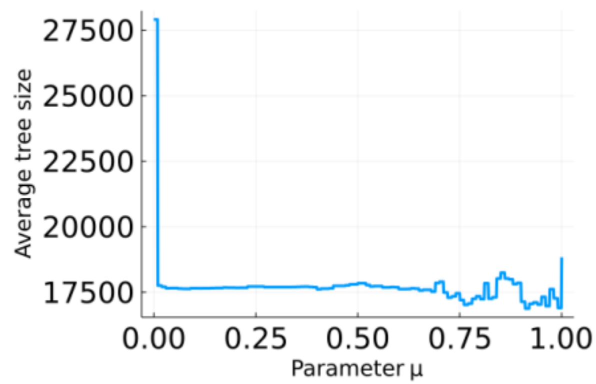


(c) $\mu \cdot \mathtt{D} + (1 - \mu) \cdot \mathtt{P}$.

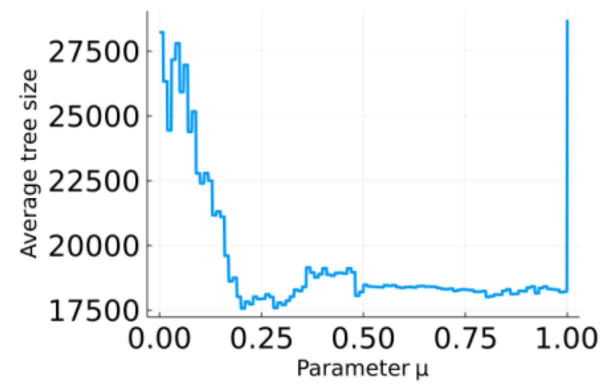
Figure 1 Chvátal distribution with 35 items and 2 knapsacks.



(a) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{P}$.



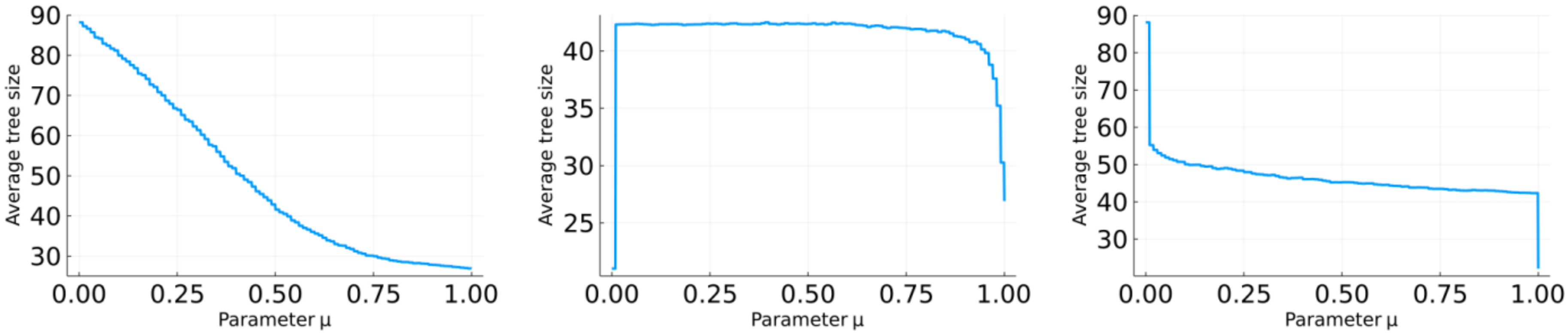
(b) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{D}$.



(c) $\mu \cdot \mathtt{D} + (1 - \mu) \cdot \mathtt{P}$.

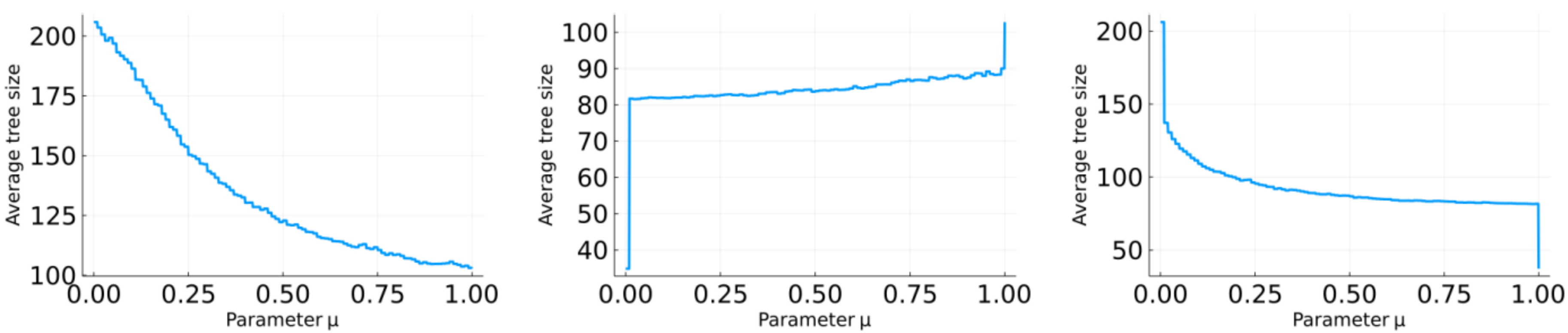
Figure 2 Chvátal distribution with 35 items and 3 knapsacks.

Knapsack cover cuts – an experiment



(a) $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1-\mu) \cdot \mathtt{P}$.

Figure 3 Reverse Chvátal distribution with 100 items and 10 knapsacks.



(a) $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1-\mu) \cdot \mathtt{P}$.

Figure 4 Reverse Chvátal distribution with 100 items and 15 knapsacks.